SOME RESULTS ON EVOLUTION

Contents

1. Introduction.

Let $z_1, \dots, z_n$ be complex coordinates in $\mathbb{C}^n$, $n \geq 2$. Given a smooth function ϱ we set

$$|\partial \varrho|^2 = \sum_{1 \leq \alpha \leq n} |\varrho_\alpha|^2,$$

$\varrho_\alpha = \varrho_{z_\alpha}, \varrho_{\bar\alpha} = \varrho_{\bar z_\alpha}, 1 \leq \alpha \leq n.$

Let M be a smooth hypersurface in $\mathbb{C}^n$ of local equation $\varrho = 0$. For every point $p \in M$ let $HT_p(M) \subset T_p(M)$ be the complex tangent hyperplane to M at p and $\nu = (\varrho_{\bar 1}, \dots, \varrho_{\bar n})$ the normal vector to $HT_p(M)$.

Let $\{E_1, \dots, E_n\}$ be an ortormal frame with origin at p and such that $\{E_1, \dots, E_{n-1}\}$ is a frame in $HT_p(M)$ and $\zeta_1, \dots, \zeta_n$ the complex coordinates determined by $\{E_1, \dots, E_n\}$. The restriction to $\{\zeta_n = 0\}$ of the Levi form of ϱ is the *intrinsec Levi form* of M at p. Its

trace is

$$\mathcal{H}(\varrho) = |\partial\varrho|^{-1} \sum_{\alpha,\beta=1}^{n} \left(\delta^{\alpha\bar{\beta}} - \frac{\varrho_\alpha \varrho_{\bar{\beta}}}{|\partial\varrho|^2} \right) \varrho_{\alpha\bar{\beta}}$$

at p.

For $n = 2$, $\mathcal{H}$ is esentially the Levi operator.

Let K be a compact subset of $\mathbb{C}^n$, $g : \mathbb{C}^n \to \mathbb{R}$ a continuous function which is constant for $|z| \gg 0$ and such that $K = \{g = 0\}$. Assume that $v \in C^0(\mathbb{C}^n \times \mathbb{R}^+)$ is a *weak solution* of the parabolic problem

$$(\star) \quad \begin{cases} v_t = \sum_{\alpha,\beta=1}^{n} \left(\delta^{\alpha\beta} - \frac{v_\alpha v_{\bar{\beta}}}{|\partial v|^2} \right) v_{\alpha\bar{\beta}} & \text{in } \Omega \times (0, +\infty) \\ v = g & \text{on } \mathbb{C}^n \times \{0\} \\ v = \text{const} & \text{for } t \gg 0. \end{cases}$$

Then the family $\{K_t\}_{t\geq 0}$ of the subsets $K_t = \{z \in \mathbb{C}^n : v(z,t) = 0\}$ (which actually depends only on K) is called the *evolution of* K *by* $\mathcal{H}$.

Evolution of a compact subset K of $\mathbb{C}^2$ was introduced in [7], [8] where, after proving that the parabolic problem has a unique (weak) solution u, it was shown that if Ω is a bounded pseudoconvex domain of $\mathbb{C}^2$ with boundary of class C^3, the evolution $\{\overline{\Omega}_t\}_{t\geq 0}$ of $\overline{\Omega}$ is contained in $\overline{\Omega}$. Conversely, pseudoconcave points "move out by evolution", i.e. if Ω is not pseudoconvex then $\overline{\Omega}_t \not\subseteq \Omega$ for some $t > 0$ (cfr. [9, Theorem 0.1]). The natural problem of what kind of hull one can recover by evolution was investigated in [11].

In this paper we consider the evolution of a compact subset of $\mathbb{C}^n$ by $\mathcal{H}$ *with a fixed part* $K^* \subseteq K$. Precisely, we study the following parabolic problem:

$$(P) \quad \begin{cases} v_t = \sum_{\alpha,\beta=1}^{n} \left(\delta^{\alpha\beta} - \frac{v_\alpha v_{\bar{\beta}}}{|\partial v|^2} \right) v_{\alpha\bar{\beta}} & \text{in } \Omega \times (0, +\infty) \\ v = g & \text{on } \overline{\Omega} \times \{0\} \\ v(z,t) = g(z) & \text{for } z \in \mathrm{b}\,\Omega \times (0, +\infty) \end{cases}$$

where Ω is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ such that

$$K \setminus K^* \subseteq \Omega, \quad K^* \subseteq \mathrm{b}\,\Omega$$

and $g : \overline{\Omega} \to \mathbb{R}$ is a continuous function such that $g^{-1}(0) = K$. In Section 2 (see Theorems 2.6, 2.7) we will prove that

a) the problem (P) has a unique (weak) solution v which is bounded and uniformly continuous in $\overline{\Omega} \times [0, +\infty)$;
b) if g is a C^2 function, the corresponding solution v of (P) is Lipschitz on $\overline{\Omega} \times [0, +\infty)$;
c) the set

$$X = \big\{(z, t) \in \overline{\Omega} \times [0, +\infty) : v(z, t) = 0\big\}$$

satisfies

$$X \cap \big(\overline{\Omega} \times \{0\}\big) = K \times \{0\}, \;\; X \cap (\mathrm{b}\,\Omega \times [0, +\infty)) = K^* \times [0, +\infty)$$

and it is actually independent of the choice of g and Ω.

The family $\{E_t(K, K^*)\}_{t \geq 0}$ of compact subsets defined by

$$E_t(K, K^*) = \{z \in \mathbb{C}^n : (z, t) \in X \text{ i.e. } v(z, t) = 0\}\,.$$

is then said to be the *evolution* of K with fixed part K^* (by $\mathcal{H}$).

Of particular interest in this setting is the case when K is the graph M of a continuous function on the closure $\overline{D}$ of a bounded domain D in $\mathbb{C}^{n-1} \times \mathbb{R}$ and $K^* = \mathrm{b}\,M$ is the boundary $\mathrm{b}\,M$ of M. Generalizing the results of [10] for $n = 2$ we then prove the following theorem (see Theorem 4.4: if D is bounded, strictly pseudoconvex domain i.e. $D \times i\mathbb{R}$ is a strictly pseudoconvex domain in $\mathbb{C}^n$ then

d) $E_t(M, \mathrm{b}\,M)$ is a graph for all $t \geq 0$ (Theorem 3.1);
e) if $\mathrm{b}\,M$ is smooth and satisfies the compatibility conditions discovered in [2], then asymptotically $E_t(M, \mathrm{b}\,M)$ approaches, in the C^0-topology the Levi flat hypersurface with boundary $\mathrm{b}\,M$ whose existence was proved in [2].

Let us mention that in the smooth case a parabolic initial value problem related to the flow of a real hypersurface of $\mathbb{C}^n$ by the trace of the Levi form is studied in a nice paper by Huisken and Klingenberg (cfr. [4]).

2. Solution of the parabolic problem.

2.1. **Geometric properties of weak solutions.** Let $U \subset \mathbb{C}^n \times (0, +\infty)$ be an open subset. An upper semicontinuous function $v : U \to [-\infty, +\infty)$ is said to be a *(weak) subsolution* of

$$v_t = \mathcal{H}(v) = \sum_{\alpha,\beta=1}^{n} (\delta^{\alpha\bar{\beta}} - |\partial v|^{-2} v_\alpha v_{\bar{\beta}}) v_{\alpha\bar{\beta}}.$$

if, for every (z^0, t^0) and a (viscosity) test function ϕ at (z^0, t^0) (i.e. ϕ is smooth near (z^0, t^0) and $v - \phi$ has a local maximum at (z^0, t^0)), one has

$$\phi_t(z^0, t^0) \leq \mathcal{H}(\phi)(z^0, t^0)$$

if $\partial\phi(z^0, t^0) \neq 0$ and

$$\phi_t(z^0, t^0) \leq \sum_{\alpha,\beta=1}^{n} \left(\delta^{\alpha\bar{\beta}} - \eta_\alpha \eta_{\bar{\beta}}\right) \phi_{\alpha\bar{\beta}}(z^0, t^0)$$

for some $\eta \in \mathbb{C}^n$ with $|\eta| \leq 1$, if $\partial\phi(z^0, t^0) = 0$.

A lower semicontinuous function $v : U \to (-\infty, +\infty]$ is said to be a *(weak) supersolution* if, for every (z^0, t^0) and a test function ϕ at (z^0, t^0) (i.e. ϕ is smooth near (z^0, t^0) and $v - \phi$ has a local minimum at (z^0, t^0)), one has

$$\phi_t(z^0, t^0) \geq \mathcal{H}(\phi)(z^0, t^0)$$

if $\partial\phi(z^0, t^0) \neq 0$ and

$$\phi_t(z^0, t^0) \geq \sum_{\alpha,\beta=1}^{n} \left(\delta^{\alpha\bar{\beta}} - \eta_\alpha \eta_{\bar{\beta}}\right) \phi_{\alpha\bar{\beta}}(z^0, t^0)$$

for some $\eta \in \mathbb{C}^n$ with $|\eta| \leq 1$, if $\partial\phi(z^0, t^0) = 0$.

Remark 2.1. Let A be an $n \times n$ hermitian matrix and $\eta \in \mathbb{C}^n$ with $|\eta| \le 1$. Then $\mathrm{Tr}A > \overline{\eta}^{\mathrm{t}} A \eta$ provided $A > 0$. Conversely, if $\mathrm{Tr}A > \overline{\eta}^{\mathrm{t}} A \eta$ for some $\eta \in \mathbb{C}^n$ with $|\eta| \le 1$ then A cannot be negative definite. In particular, from the above definition it follows that plurisubharmonic functions are (weak) subsolutions to $v_t = \mathcal{H}(v)$.

A *(weak) solution* is a continuous function which is both a subsolution and a supersolution.

One checks that the following properties are true:

1) maximum (minimum) of a finite number of subsolutions (supersolutions) is a subsolution (supersolution);
2) if $W' \subset W \subset \mathbb{C}^n \times (0, +\infty)$, W, W' open and $v : W \to (-\infty, +\infty)$, $v' : W' \to [-\infty, +\infty)$ are subsolutions, such that for all $\zeta \in bW' \cap W$
$$\limsup_{z \to \zeta} v'(z) \le v(\zeta)$$
then the function
$$w(z) = \begin{cases} \max\,(v(z), v'(z)) & \text{if } z \in W' \\ v(z) & \text{if } z \in W \setminus W' \end{cases}$$
is a subsolution in W;
3) translations of subsolutions (supersolutions) are subsolutions (supersolutions); i.e. if $\zeta \in \mathbb{C}^n$, $h \in \mathbb{R}$ is positive and $v^{\zeta,h}(z,t) = v(z+\zeta, t+h)$ then $v^{\zeta,h}$ is a subsolution (supersolution) provided v is;
4) the limit of a decreasing sequence of subsolutions is a subsolution.

Lemma 2.1. *If $\varrho : (a,b) \to \mathbb{R}$ is a continuous non decreasing function and v is a subsolution (or a supersolution) with the range of v in (a,b), then $\varrho \circ v$ is a subsolution (or a supersolution, respectively). In particular, if v is a weak solution, then $\varrho \circ v$ is a solution.*

Proof. There is a sequence of C^∞ functions $\varrho_n : (a,b) \to \mathbb{R}$ such that $\varrho_n'(t) > 0$, $\varrho_n(t) \searrow \varrho(t)$, $t \in (a,b)$, therefore it suffices to prove the lemma for $\varrho = \varrho_n$, for then $\varrho_n \circ v \searrow \varrho \circ v$

and $\varrho \circ v$ will be a subsolution due to 4). Let ϕ be a test function for $\varrho \circ v$. Then $\chi = \varrho^{-1}$ is smooth and (strictly) increasing; since $\psi \chi \circ \phi$ is a test function for v hence we have

$$\psi_t(z^0, t^0) \leq \mathcal{H}(\psi)(z^0, t^0)$$

if $\partial\psi(z^0, t^0) \neq 0$ and

$$\psi_t(z^0, t^0) \leq \sum_{\alpha,\beta=1}^{n} \left(\delta^{\alpha\bar{\beta}} - \eta_\alpha \eta_{\bar{\beta}}\right) \psi_{\alpha\bar{\beta}}(z^0, t^0)$$

for some $\eta \in \mathbb{C}^n$ with $|\eta| \leq 1$, if $\partial\psi(z^0, t^0) = 0$.

Consider now the case $\partial\psi(z^0, t^0) \neq 0$ and suppose, by a contradiction, that

$$\phi_t(z^0, t^0) > \mathcal{H}(\phi)(z^0, t^0).$$

Then

$$\begin{aligned} \psi_t(z^0, t^0) &= \chi'(\phi(z^0, t^0))\phi_t(z^0, t^0) \\ &> \chi'(\phi(z^0, t^0))\mathcal{H}(\phi)(z^0, t^0) = \mathcal{H}(\psi)(z^0, t^0) \end{aligned}$$

which is absurd.

As for the case $\partial\psi(z^0, t^0) = 0$ it is enough to show the following: let $W \subset \mathbb{C}^n$ be open and $\varrho : W \to \mathbb{R}$ a weak continuous solution of the inequality

$$\mathcal{H}(\varrho)(z) \geq -h(z)$$

where $h : W \to \mathbb{R}^+$ is a continuous positive function. Suppose that χ is a continuous increasing function $\mathbb{R} \to \mathbb{R}$ with $\chi' \in L^\infty(\mathbb{R})$ and $0 \leq \chi' \leq 1$. Then

$$H(\chi \circ \varrho)(z) \geq -h(z),$$

in the weak sense. We proceed as follows. Since χ can be approximated uniformly on compact subsets of $\mathbb{R}$ by smooth functions with the required properties, we may assume

that $\chi : \mathbb{R} \to \mathbb{R}$, $\chi \in C^\infty(\mathbb{R})$, $0 < \chi'(s) \leq 1$; hence $\chi^{-1} \in C^\infty(\mathbb{R})$. Let ψ be a smooth test function for $\mathcal{H}(\chi \circ \varrho) \geq -h$, i.e.

$$\psi(z) \geq (\chi \circ \varrho)(z) \text{ and } \psi(z^0) = (\chi \circ \varrho)(z^0);$$

then $\psi^* = \chi^{-1} \circ \varrho$ is a test function too, i.e.

$$\psi^\star(z) \geq \varrho(z), \psi^*(z^0) = \varrho(z^0).$$

If $\partial\psi(z^0) \neq 0$ we have $\partial\psi^*(z^0) \neq 0$ and, by virtue of the hypothesis, $\mathcal{H}(\psi^*)(z^0) \geq -h(z^0)$, hence

$$\begin{aligned} \mathcal{H}(\psi)(z) = \mathcal{H}(\chi \circ \psi^*)(z^0) &= \chi'(\psi^*(z^0))\mathcal{H}(\psi^\star)(z^0) \geq \\ &\geq -\chi'(\psi^*(z^0))h(z^0) > -h(z^0). \end{aligned}$$

If $\partial\psi(z^0) = 0$, then $\partial\psi^*(z^0) = 0$ and there is a vector $\eta \in \mathbb{C}^n$, $|\eta| \leq 1$, with

$$\sum_{\alpha,\beta=1}^{n} \left(\delta^{\alpha\bar\beta} - \eta_\alpha\eta_{\bar\beta}\right) \phi_{\alpha\bar\beta}(z^0, t^0) \geq -h(z^0).$$

Now we observe that, since $\psi^*_\alpha(z^0) = 0$, $1 \leq \alpha \leq n$

$$\begin{aligned} \sum_{\alpha,\beta=1}^{n} \left(\delta^{\alpha\bar\beta} - \eta_\alpha\eta_{\bar\beta}\right) \phi_{\alpha\bar\beta}(z^0, t^0) &= \chi'(\psi^*(z^0))\phi_t(z^0, t^0) \sum_{\alpha,\beta=1}^{n} \left(\delta^{\alpha\bar\beta} - \eta_\alpha\eta_{\bar\beta}\right) \phi_{\alpha\bar\beta}(z^0, t^0) \geq \\ &\quad -\chi'(\psi^*(z^0))h(z^0) \geq -h(z^0). \end{aligned}$$

This ends the proof. $\square$

In the sequel we will use the following

Proposition 2.2. *Let $\{v_\alpha\}_{\alpha\in A}$ be a family of weak subsolution of $v_t = \mathcal{H}(v)$ and assume that $v = \sup_{\alpha\in A} v_\alpha$ is locally bounded from above. Then the upper semicontinuous regularization of v*

$$v^*(z, t) = \limsup_{(z',t')\to(z,t)} v(z', t').$$

is a weak subsolution.

Proof. We first prove the following: let $B \Subset W$ be a ball of radius r centered at $w^0 = (z^0, t^0)$ and ϕ be such that $(v-\phi)(w^0) > (v-\phi)(w)$ for $w \in \overline{B} \setminus w^0$. Then there is a sequence $w^\nu \to w^0$ and indices $\alpha_\nu \in A$ such that for every ν the function $v_{\alpha_\nu} - \phi$ has a maximum at w^ν (relative to $\overline{B}$).

We may assume that $(v-\phi)(w^0) = 0$. For every $\nu \in \mathbb{N}$ such that $1/\nu \le r$ let

$$-\delta_\nu = \max\left\{(v-\phi)(w) : 1/\nu \le r|w-w^0| \le r\right\}.$$

Since $v-\phi$ has a strict maximum (=0) at w^0 (relative to $\overline{B}$), $-\delta_\nu < 0$ i.e. $\delta_\nu > 0$. By definition of regularization

$$\Big\{(w,s) \in B \times [-\infty, +\infty) : s \le (v^* - \phi)(w)\Big\}$$

is the closure of

$$\bigcup_{\alpha \in A} \left\{(w,s) \in B \times [-\infty, +\infty) : s \le (v_\alpha - \phi)(w)\right\}.$$

Thus, for every ν there is a point $(w^\nu, s^\nu) \in B \times \mathbb{R}$ and $\alpha_\nu \in A$ such that

$$s^\nu \le (v_{\alpha_\nu} - \phi)(w^\nu) \le 0, \;\; |w^\nu - w^0| + s^\nu \le \frac{1}{2}\min(\delta_\nu, 1/\nu);$$

in particular

$$|w^\nu - w^0| \le \frac{1}{\nu}, \;\; -\frac{1}{2}\delta_\nu (v_{\alpha_\nu} - \phi)(w^\nu) \le 0.$$

Let now w^ν denote any of the maximum points of $(v_{\alpha_\nu} - \phi)_{|\overline{B}}$. Since

$$\begin{aligned}
(v_{\alpha_\nu} - \phi)(w^\nu) &\ge -\frac{1}{2}\delta_\nu > -\delta_\nu \\
&\ge \max\Big\{(v-\phi)(w) : \nu^{-1} \le |w-w^0| \le r\Big\} \\
&> \max\Big\{(v_{\alpha_\nu} - \phi)(w) : \nu^{-1} \le |w-w^0| \le r\Big\}
\end{aligned}$$

we conclude that $|w^\nu - w^0| \le \nu^{-1}$ i.e. $w^\nu \to w^0$.

In order to prove that v^* is a weak subsolution let $\phi \in C^\infty(B)$ and suppose that $v^* - \phi$ has a maximum at $w^0 = (z^0, t^0)$ with $\partial\phi(z^0, t^0) \ne 0$. Let $\phi_\varepsilon(w) = \phi(w) + \varepsilon|w - w^0|^2$; then $\partial\phi(z^0, t^0) \ne 0$, ϕ_ε has a strict maximum at w^0 so, in view of what already proved, there

are point $w^\nu = (z^\nu, t^\nu) \to w^0 = (z^0, t^0)$ and $\alpha_\nu \in A$ such that $(v_{\alpha_\nu} - \phi_\varepsilon)$ have maximum at w^ν with $\partial\phi_\varepsilon(z^\nu, t^n) \neq 0$ and

$$\mathcal{H}(\phi_\varepsilon)(z^\nu, t^\nu) \geq \frac{\partial \phi_\varepsilon}{\partial t}(z^\nu, t^\nu).$$

Letting $\nu \to +\infty$, we get

$$\mathcal{H}(\phi_\varepsilon)(z^0, t^0) \geq \frac{\partial \phi_\varepsilon}{\partial t}(z^0, t^0)$$

and then with $\varepsilon \to 0$

$$\mathcal{H}(\phi_\varepsilon)(z^0, t^0) \geq \frac{\partial \phi_\varepsilon}{\partial t}(z^0, t^0).$$

The proof if $\partial\phi(z^0, t^0) = 0$ is similar. $\square$

Finally, in order to prove the independence of the evolution of the pair (K, K^*) on Ω (see Introduction, c)) we discuss a local maximum property of the level sets of a weak solution v.

For an open set V in $\mathbb{C}^n \times (0, +\infty)$ set

$$\mathcal{P}_\mathcal{H}(V) = \Big\{\psi \in C^2(V) : \psi_t \leq \mathcal{H}(\psi)\Big\}.$$

Let Z be a locally closed subset of V. We say that Z has *local maximum property* (*relative to* $\mathcal{P}_\mathcal{H}$) if for every open set $V \Subset \mathbb{C}^n \times (0, +\infty)$ such that $\overline{V} \cap Z$ is closed and $\overline{V}$ is compact, and for every $\psi \in \mathcal{P}_\mathcal{H}(V')$ where V' is a neighbourhood of $\overline{V}$ it holds:

$$\max_{\overline{V} \cap Z} \psi = \max_{\mathrm{b} V \cap Z} \psi.$$

Lemma 2.3. *Let $W \subseteq \mathbb{C}^n \times (0, +\infty)$ be open, $v : W \to \mathbb{R}$ a weak solution of the $v_t = \mathcal{H}(v)$ and $Z = \{v = 0\}$. Then*

a) *Z has local maximum property;*

b) *for every $c > 0$, $Z^c = \big\{(z, t) \in Z : t \leq c\big\}$ has local maximum property.*

Proof. We first prove the following. Let v be a weak supersolution of $v_t = \mathcal{H}(v)$ in W. Fix a point $(z^0, t^0) \in W$ and a neighbourhood $V \subset W$ of (z^0, t^0). Let $\phi \in C^2(V)$ be such that $\phi(z^0, t^0) = v(z^0, t^0) = c$ and

$$\Big\{(z,t) \in V : \phi(z,t) > c\Big\} \subseteq \Big\{(z,t) \in V : v(z,t) > c\Big\}. \tag{1}$$

Then

$$\phi_t(z^0, t^0) \geq \mathcal{H}(\phi)(z^0, t^0)$$

if $\partial\phi(z^0, t^0) \neq 0$ and

$$\phi_t(z^0, t^0) \geq \sum_{\alpha,\beta=1}^{n} \Big(\delta^{\alpha\bar{\beta}} - \eta_\alpha \eta_{\bar{\beta}}\Big)\, \phi_{\alpha\bar{\beta}}(z^0, t^0)$$

for some $\eta \in \mathbb{C}^n$ with $|\eta| \leq 1$, if $\partial\phi(z^0, t^0) = 0$.

Observe that, if there exists a non-decreasing continuous function $\varrho : \mathbb{R} \to \mathbb{R}$ such that $\varrho(c) = c$ and $\phi(z,t) \leq (\varrho \circ u)(z,t)$ on a neighbourhood of (z^0, t^0), then $\varrho \circ u$ is still a weak supersolution, so the conclusions concerning ϕ are immediate.

In order to construct ϱ let N be a compact neighbourhood of (z^0, t^0) such that $N \subset V \subset W$. Set $\varrho_1(s) = c$ for $s \leq c$. For every s satisfying

$$c \leq s \leq s_\infty := \sup \Big\{ v(z,t) : (z,t) \in N Big\}$$

let

$$R_s = \Big\{(z,t) : (z,t) \in N : v(z,t) \leq s\Big\}.$$

Since v is lower semicontinuous, the R_s's are compact and $R_s \subset R_{s'}$ if $s \leq s'$. For $c \leq s \leq s_\infty$ we then define

$$\varrho_1(s) = \max\{\phi(z,t) : (z,t) \in R_s\}.$$

Clearly, ϱ_1 is a non decreasing upper semicontinuous function, $s \mapsto R_s$ being an upper semicontinuous correspondence. Moreover, $\phi(z,t) \leq (\varrho \circ u)(z,t)$. Indeed, assume for a contradiction that $\phi(z,t) > (\varrho \circ u)(z,t)$. If $\phi(z,t) > (\varrho \circ u)(z,t)$ this is impossible as

$\varrho_1 \geq c$ always. If $\phi(z,t) > c$, by 1, $v(z,t) > c$. Let $s = v(z,t)$; then $(z,t) \in R_s$ and so $\varrho_1(s) \geq \phi(z,t)$, i.e. $\phi(z,t) \leq (\varrho \circ u)(z,t)$. Choose finally a continuous non decreasing function $\varrho : \mathbb{R} \to \mathbb{R}$ such that $\varrho \geq \varrho_1$, $\varrho(c) = c$. Then $\phi(z,t) \leq (\varrho \circ u)(z,t)$. (Note that ϱ can be chosen continuous because $\lim\limits_{s\to 0^+} \varrho_1(s) = c$).

Now suppose the claim a) is false, i.e.

$$\max_{\overline{V}\cap Z} \psi > \max_{\mathrm{b}\,V\cap Z} \psi,$$

for some $\psi \in \mathcal{P}_{\mathcal{H}}(V')$. Then there is $\varepsilon > 0$ small enough so that the function $\psi^\varepsilon = \psi - \varepsilon t$ still satsfies

$$\max_{\overline{V}\cap Z} \psi^\varepsilon > \max_{bV\cap Z} \psi^\varepsilon,$$

and, in addition $\psi^\varepsilon_t < \mathcal{H}(\psi^\varepsilon)$ in $\overline{V}$. Let (z^0,t^0) denote the point where ψ^ε takes maximum value, say M, relative to $\overline{V} \cap Z$. Clearly $(z^0,t^0) \in V \cap Z$, and

$$\begin{aligned}\Big\{(z,t) \in V : \psi^\varepsilon(z,t) > m\Big\} \subset V \setminus Z \;&=\; \Big\{(z,t) \in V : u(z,t) \neq 0\Big\}\\ &=\; \Big\{(z,t) \in V : u(z,t)^2 > 0\Big\}.\end{aligned}$$

If we set $\phi = \psi^\varepsilon - m$ and $w = u^2$, then w is still a weak solution of the parabolic problem, $\phi(z^0,t^0) = w(z^0,t^0)$ and

$$\{(z,t) \in V : \phi(z,t) > 0\} \subset \{(z,t) \in V : w(z,t) > 0\}.$$

Taking into account what proved in the first part we obtain

$$\psi^\varepsilon_t(z^0,t^0) = \phi_t(z^0,t^0) \geq \mathcal{H}(\phi)(z^0,t^0) = \mathcal{H}(\psi^\varepsilon)(z^0,t^0)$$

which is a contradiction.

In order to prove b) fix $c > 0$ and consider ψ as in definition of local maximum property. Let $\varrho : \mathbb{R} \to \mathbb{R}$ defined by $\varrho(t) = 0$ if $t \leq c$, $\varrho(t) = -(c-t)^3$ if $t > c$ and, for $N > 0$, $(z,t) \in \overline{V}$, let $\psi^N(z,t) = \psi(z,t) + N\varrho(t)$. Clearly $\psi^N \in \mathcal{P}_{\mathcal{H}}(V')$ and so, by part

a),

$$\max_{\overline{V}\cap X} \psi^N = \max_{\mathrm{b}\,V\cap X} \psi^N.$$

Observe, however, that

$$\lim_{N\to+\infty} \psi^N(z,t) = -\infty$$

if $t > c$ and

$$\psi(z,t)^N = \psi(z,t)$$

for $(z,t) \in X^c$, thus

$$\lim_{N\to+\infty} \max_{\overline{V}\cap X} \psi^N = \max_{\overline{V}\cap X^c} \psi.$$

The same being true for $\mathrm{b}\,V \cap X^c$, we conclude that

$$\max_{\overline{V}\cap X^c} \psi^N = \max_{\mathrm{b}\,V\cap X^c} \psi^N.$$

□

2.2. **Comparison principle. Walsh Lemma in unbounded domains.** Let us consider the cylinder $Q = \Omega \times (0,h)$ in $\mathbb{C}^n \times \overline{\mathbb{R}}^+$, where Ω is a bounded domain of $\mathbb{C}^n$ and let

$$\Sigma = (\overline{\Omega} \times \{0\}) \cup (b\Omega \times (0,h)).$$

We have the following comparison principle which can be proved arguing as in [8, Theorem 1.1].

Theorem 2.4. *Let $v, w \in C^0(\overline{Q})$ be respectively a weak subsolution and a weak supersolution in Q. If $v \le w$ on $\overline{\Sigma}$ then $v \le w$. In particular, $v \le \max_{\overline{\Sigma}} v$, $w \ge \min_{\overline{\Sigma}} w$.*

We also need the following unbounded version of the Walsh Lemma proved in [11].

Let $W \subset \mathbb{R}^N$ be a domain with $\mathrm{b}\,W \neq \varnothing$ and $\mathcal{F} = \mathcal{F}(\overline{W})$ a class of functions v satisfying the following properties:

1) v is upper semicontinuous in $\overline{W}$ and $\sup_{\overline{W}} v = \sup_{\mathrm{b}\,W} v < +\infty$;

2) for every constant α, $v+\alpha \in \mathcal{F}$, if $v \in \mathcal{F}$;

3) if v is locally equal to the maximum of finitely many translates of functions in $\mathcal{F}$, then $v \in \mathcal{F}$;

4) upper semicontinuous regularization of the supremum of a family of functions in $\mathcal{F}$ is a member of $\mathcal{F}$, provided it is uniformly bounded on $\overline{W}$;

5) if W' is relatively open in $\overline{W}$, $v \in \mathcal{F}(\overline{W})$, $v' \in \mathcal{F}(\overline{W}')$ and $v'(\zeta) \leq v(\zeta)$ for $\zeta \in (\mathrm{b}\, W') \cap W$ then the function

$$w(z) = \begin{cases} \max\,(v(z), v'(z)) & \text{if } z \in W' \\ v(z) & \text{if } z \in W \end{cases}$$

belongs to $\mathcal{F}(\overline{W})$.

Lemma 2.5. *Let $g \in C^0(\mathrm{b}\, W)$ be a bounded uniformly continuous function and*

$$v(z) = \sup\Big\{w(z) : v \in \mathcal{F}, w \leq g \text{ on } \mathrm{b}\, W\Big\}.$$

Suppose that $v = g$ on $\mathrm{b}\, W$ and v is uniformly continuous at the points of $\mathrm{b}\, W$, with the modulus of continuity $\omega(\delta)$, $\lim_{\delta\to 0^+}\omega(\delta) = 0$, i.e.

$$\sup\,\Big\{|v(z) - g(\zeta)|, \zeta \in \mathrm{b}\, W, z \in \overline{W}, |z-\zeta| \leq \delta\Big\} \leq \omega(\delta).$$

Then v is uniformly continuous on $\overline{W}$ with the same $\omega(\delta)$ as its modulus of continuity.

2.3. **Existence of solutions and evolution.** We are in position to prove the following existence theorem:

Theorem 2.6. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, $g : \overline{\Omega} \to \mathbb{R}$ a continuous function. Then the problem (P) has a unique weak solution v which is bounded and uniformly continuous in $\overline{\Omega} \times [0, +\infty)$.*

Proof. Unicity is a consequence of the comparison principle. Existence will be proved by Perron method.

Let $W = \Omega \times [0, +\infty)$ and $\mathcal{F} = \mathcal{F}_{\mathrm{g}}$ be the class of all functions $w : \overline{W} \to [-\infty, +\infty)$ with the following properties:

1) w is upper semicontinuous in $\overline{W}$ and is a subsolution in W;

2) $w \leq \max\limits_{\overline{W}} g$;

3) $w \leq g$ on $\mathrm{b}\, W$

Let $v : \overline{W} \to \mathbb{R}$ be the function

$$(z,t) \longrightarrow \sup \Big\{ w(z,t) : w \in \mathcal{F} \Big\}$$

and v^* its upper semicontinuous regularization: v^* is a subsolution (cfr. Proposition 2.2).

We want to prove that $v = v^*$ and v is actually the solution of the problem (P).

The proof is divided in several steps.

A) $v = g$, for $(\zeta, t) \in \mathrm{b}\,\Omega \times [0, +\infty)$. Furthermore, v is uniformly continuous at the points of $\mathrm{b}\,\Omega \times [0, +\infty)$ in the following sense: given $\varepsilon > 0$ there exists $\delta > 0$ such that $|v(z,t) - g| \leq \varepsilon$ if $\operatorname{dist}((z,t), \mathrm{b}\,\Omega \times [0, +\infty)) < \delta$.

Let $\varepsilon > 0$ be fixed and $g_1 \in C^2(\mathbb{C}^n)$ such that $|g_1 - g| < \varepsilon$ on $\overline{\Omega}$. Since Ω is strictly pseudoconvex there is a strictly plurisubharmonic function ϱ on a neighbourhood U of $\overline{\Omega}$, such that $\varrho = 0$ on $\mathrm{b}\,\Omega$ and $\Omega = \{\varrho < 0\}$. For $m > 0$ big enough the time-independent function

$$v_m^\varepsilon(z) = m\varrho(z) + g_1 - \varepsilon,$$

is strongly plurisubharmonic in U, therefore a subsolution of $v_t = \mathcal{H}(v)$ and

$$v_m^\varepsilon(z) = n\varrho(z) + g_1 - \varepsilon \leq n\varrho(z) + g \leq g(z)$$

for $z \in \overline{W}$..

Thus $v_m^\varepsilon \in \mathcal{F}$ and consequently

$$m\varrho(z) + g - \varepsilon \leq v_m^\varepsilon(z,t) \leq v(z,t) \leq g.$$

It follows

$$|v(z,t) - g| \leq m|\varrho| + \varepsilon.$$

It is evident now that, for a fixed ε, there is $\delta > 0$ such that the statement A) holds true.

B) For all $a \in \Omega$

$$\lim_{(z,t)\to(a,0)} v(z,t) = \lim_{(z,t)\to(a,0)} v^*(z,t) = g(a) \tag{2}$$

In order to prove this we fix $\varepsilon > 0$ and smooth functions ϕ, ψ on $\mathbb{C}^n$ in such a way to have

$$g(z) - \varepsilon < \phi(z) < g(z) < \psi(z) < g(z) + \varepsilon.$$

Let c be a constant such that

$$\left|\sum_{\alpha,\beta=1}^{n} \phi_{\alpha\bar\beta}(z)\xi^\alpha\overline{\xi^\beta}\right| < c|\xi|^2, \left|\sum_{\alpha,\beta=1}^{n} \psi_{\alpha\bar\beta}(z)\xi^\alpha\overline{\xi^\beta}\right| < c|\xi|^2$$

for all $z \in \Omega$ and $\xi \in \mathbb{C}^n$. Then

$$v_+(z,t) = \psi(z) + ct, \;\; v_-(z,t) = \phi(z,t) - ct$$

are respectively a regular supersolution and a regular subsolution in W; moreover, $v_- \in \mathcal{F}$ and $v_{|\mathrm{b}\,W} \leq v_{+|\mathrm{b}\,W}$. In view of the comparison principle for v_+ and v_-, we deduce that

$$v_- \leq v \leq v^* \leq v_+$$

in $\overline{W}$ and consequently, since v_- and v_+ are continuous, that

$$g(a) - \varepsilon \leq v_-(a,0) \leq \liminf_{(z,t)\to(a,0)} v(z,t) \leq \limsup_{(z,t)\to(a,0)} v(z,t) \leq u_+(a,0) \leq g(a) + \varepsilon.$$

for all $a \in \overline{\Omega}$.

2 follows ε being arbitrary.

A), B) imply that $v^* \in \mathcal{F}$ therefore, by definition of v, we have $v^* = v$. In particular, $v = v^*$ is a subsolution which is continuous at every point of $\mathrm{b}\,W$. Thus all the hypothesis of the Walsh Lemma (see 2.5) are satisfied hence v is continuous in $\overline{W}$.

Finally v is a weak solution in W. For if not there is $(z^0, t^0) \in W$ and $\phi \in C^\infty(W)$ such that $v - \phi$ has a strict local minimum (=0) at (z^0, t^0) and

$$\phi_t(z^0, t^0) < \mathcal{H}(\phi)(z^0, t^0)$$

if $\partial\phi(z^0, t^0) \neq 0$ and

$$\phi_t(z^0, t^0) < \sum_{\alpha,\beta=1}^{n} \left(\delta^{\alpha\bar{\beta}} - \eta_\alpha \eta_{\bar{\beta}}\right) \phi_{\alpha\bar{\beta}}(z^0, t^0) \tag{3}$$

for some $\eta \in \mathbb{C}^n$ with $|\eta| \leq 1$, if $\partial\phi(z^0, t^0) = 0$. Observe that $v(z^0, t^0) < \max_{\overline{W}} g$, otherwise, by definition of v, (z^0, t^0) would be a maximum point for v hence for ϕ and this contadicts 3 (see Remark 2.1). Thus, we can find $\varepsilon > 0$ small enough such that $\phi + \varepsilon$ is a subsolution on a neighbourhood U of (z^0, t^0), $\phi + \varepsilon < \max_{\overline{W}} g$ and

$$\varnothing \neq V = \Big\{(z,t) \in U : (\phi + \varepsilon - v)(z,t) > 0\Big\} \Subset U.$$

It is now clear that

$$\tilde{v}(z,t) = \begin{cases} \max\left(v(z,t), \phi(z,t) + \varepsilon\right) & \text{if } (z,t) \in U \\ v(z,t) & \text{if } (z,t) \in \overline{W} \setminus \overline{V} \end{cases}$$

is a subsolution, $\tilde{v} \in \mathcal{F}$ and $v < \tilde{v}$ near (z^0, t^0): contradiction.

Theorem 2.6 is completely proved. $\square$

Remark 2.2. The strict pseudoconvexity condition can be relaxed. In particular the following condition suffices: for all $\zeta \in \mathrm{b}\Omega$ there is a ball B centered at ζ and a strictly plurisubharmonic function $\phi : B \to \mathbb{R}$ such that $\phi(\zeta) = 0$ and $\phi < 0$ on $B \cap \Omega$.

Remark 2.3. Using the method employed in [5] it can be proved that if the boundary value g is $C^2(\mathrm{b}\,\Omega)$ the solution of the problem (P) is Lipschitz in $\overline{\Omega}$.

Theorem 2.7. *Let (K^*, K) be a pair of compact sets in $\mathbb{C}^n$ such that $K^* \subset K$, $K \not\equiv \varnothing$ and Ω a bounded strictly pseudoconvex domain such that $K \setminus K^* \subseteq \Omega$, $K^* \subseteq \mathrm{b}\,\Omega$. Assume*

that $K = g^{-1}(0) = K$ *with* $g : \overline{\Omega} \to \mathbb{R}$ *and let* v *be the solution of the parabolic problem* (P)*. Then the set*

$$X = \left\{(z,t) \in \overline{\Omega} \times [0,+\infty) : v(z,t) = 0\right\}$$

is independent of the choice of g *and* Ω*. Moreover*

i) $X \cap \left(\overline{\Omega} \times \{0\}\right) = K \times \{0\}$,

ii) $X \cap (\mathrm{b}\,\Omega \times [0,+\infty)) = K^* \times [0,+\infty)$.

Proof. The independence of the zero set $\{u = 0\}$ of the choice of g satisfying $g^{-1}(0) = K$ is essentially the argument of Evans and Spruck in [3] (cfr. also [11]).

It remains to show independence of $X = u^{-1}(0)$ of the choice of Ω satisfying the conditions of Theorem 2.7 .

Suppose Ω_1, Ω_2 are such domains and $\Omega_0 = \Omega_1 \cap \Omega_2$. Then Ω_0 satisfies condition (C) of Remark 2.2 and also $K \setminus K^* \subseteq \Omega_0$, $K^* \subseteq \mathrm{b}\,\Omega_0$. For each of these sets we have unique (independent of respective u) "evolution hypersurface" i.e X_j, where $j = 0,1,2$, $X_j \subseteq \overline{\Omega}_j \times [0,+\infty)$ and

i) $X_j \cap \left(\overline{\Omega} \times \{0\}\right) = K \times \{0\}$,

ii) $X_j \cap (\mathrm{b}\,\Omega \times [0,+\infty)) = K^* \times [0,+\infty)$.

We will show that $X_1 = X_0$ and this will imply that $X_1 = X_2$, as required.

Let g, v be as in Theorem 2.7, for the domain Ω_1, so that $X_1 = u^{-1}(0)$. Let now $g_0 = g_{|\overline{\Omega}_0}$ and $u_0 \in C^0\left(\overline{\Omega}_0 \times [0,+\infty)\right)$ be the corresponding solutions of the parabolic problem so that $X_0 = u_0^{-1}(0)$.

The following is true:

i) $X_1 \subseteq \overline{\Omega}_0 \times [0,+\infty)$;

ii) $X_1 \subseteq (\Omega_0 \cup K^*) \times [0,+\infty)$.

Since Ω_0 is the intersection of two strictly pseudoconvex domains Ω_1, Ω_2, there is a neighbourhood N of $\overline{\Omega}_0$ and a continuous plurisubharmonic function $\phi : N \to \mathbb{R}$ such

that $\overline{\Omega}_0 = \{\phi \le 0\}$. Suppose $X_1 \not\subseteq \overline{\Omega}_0$, then there exists $c > 0$ such that $X^c \not\subseteq \overline{\Omega}_0$ but $X_1^c \subset N$. (Observe that $c \mapsto X^c$ is an upper semicontinuous correspondence and $X_1^\circ = K \subset N$.)

Let $\widetilde{\phi}(z,t) = \phi(z)$ and define

$$M = \max_{X^c} \widetilde{\phi}, \ \ F = \left\{(z,t) \in X^c : \widetilde{\phi}(z,t) = M\right\}.$$

Then $M > 0$, F is compact and $F \cap (K^* \times \{0\}) = \varnothing$. Choose V, a neighbourhood of F such that $\overline{V}$ is compact, $\overline{V} \subset N \setminus \overline{\Omega}_0 \times (0,+\infty)$. Then

$$M = \max_{X_1^c \cap \overline{V}} \tilde{\phi} > \max_{X_1^c \cap \mathrm{b}\, V} \tilde{\phi}$$

which contradicts the local maximum property (b) of Lemma 2.3 since, clearly, $u \in \mathcal{P}_{\mathcal{H}}$. . This proves i).

As for ii) suppose $(z^0,t^0) \in X_1 \cap (\mathrm{b}\,\Omega_0 \setminus K^*) \times [0,+\infty)$. Then $z^* \in \mathrm{b}\,\Omega_1$ or $z^* \in \mathrm{b}\,\Omega_2$. In either case there is a C^2 strictly plurisubharmonic function $v = v(z)$ in a neighbourhood of z^* such that $v(z^*) = 0$, $v(z) < 0$ for $z \in B(z^*,r) \cap (\overline{\Omega}_0 \setminus \{z^*\})$. Since v is strictly plurisubharmonic, there is an $\varepsilon > 0$, small enough so that the function $\psi^\varepsilon(z,t) = v(z) - \varepsilon(t-t^*)^2$ is of the class $\mathcal{P}_{\mathcal{H}}$ in $V = B \times (t^0 - r, t^* + r)$. Observe now that $\psi^*(z^0,t^0) = 0$ while $\psi^\varepsilon(z,t) < 0$ for $(z,t) \in X_1 \cap V \setminus \{(z^0,t^0)\}$. This contradicts again the local maximum property (a) of Lemma 2.3. Thus $X_1 \cap (\mathrm{b}\,\Omega_0 \setminus K^*) \times [0,+\infty) = \varnothing$., whence ii).

We can now show that $X_0 = X_1$. Fix $c > 0$ and let

$$W^c = \Omega_0 \times (0,c), \ \ \Sigma^c = \left(\overline{\Omega}_0 \times \{0\}\right) \cup (b\Omega_0 \times [0,c]) \, .$$

Let $U^c = u_{|\overline{W}^c}$. Then u_0, U^c are continuous weak solutions in $\overline{W}$. By i), ii)

$$u^{-1}(0) \cap \Sigma^c = (U^c)^{-1}\,(0) \cap \Sigma^c.$$

Hence, similarly as in [ES1] there are continuous increasing functions $\chi_1, \chi_2 : \mathbb{R} \to \mathbb{R}$, with $\chi_j(0) = 0$, $j = 1, 2$, such that

$$\chi_1 \circ u_0 \le U^c \le \chi_2 \circ u_0$$

on Σ^c.

Since $\chi_j \circ u_0$, $j = 1, 2$, are weak solutions the comparison principle implies that

$$\chi_1 \circ u_0 \le U^c \le \chi_2 \circ u_0$$

in W and so

$$X_0^c = (u_0)^{-1}(0) = (U^c)^{-1}(0) = X_1^c,$$

for every $c > 0$. Thus $X_0 = X_1$. $\square$

In light of this theorem we define

$$E_t(K, K^*) = \Big\{ z \in \mathbb{C}^n : (z, t) \in X \text{ i.e. } u(z, t) = 0 \Big\}.$$

The family $\{E_t(K, K^*)\}_{t \ge 0}$ is said to be the *evolution* of K mod K^* (*by* $\mathcal{H}$).

The semigroup property

$$E_{t+t'}(K, K^*) = E_t\left(E_{t'}(K, K^*), K^*\right) \tag{4}$$

holds true as well as for the standard evolution (i.e. when $K^* = \varnothing$).

2.4. **Some geometric properties.**

Theorem 2.8. *Let Ω be a bounded, strictly pseudoconvex domain of $\mathbb{C}^n$, $K \subset \overline{\Omega}$, $K^* \subset \mathrm{b}\,\Omega$ compact sets such that: $K^* \subset K$, $K \setminus K^* \subset \Omega$ and separates Ω. Let $\big\{E_t(K, K^*)\big\}_{t \ge 0}$ be the evolution of K* mod *K^*. Then, for every t the subset $E_t(K, K^*) \setminus K^*$ separates Ω.*

Proof. Choose $g \in C^0(\overline{\Omega})$ such that $g^{-1}(0) = K$; $\Omega \setminus K = \{g > 0\} \cup \{g < 0\}$ and we choose ζ_1, ζ_2 such that $g(\zeta_1) > 0$, $g(\zeta_2) < 0$. Let u be the weak solution of (P). Then

$E_t(K,K^*) = \{u(\cdot,t)=0\}$ and $\Omega \setminus \{E_t(K,K^*)\}_{t\geq 0}$ is a union $\{u(\cdot,t)>0\} \cup \{u(\cdot,t)<0\}$ of nonempty subsets. $\square$

Proposition 2.9. *In the context of the previous theorem*

$$\limsup_{t\to+\infty} E_t(K,K^*) = K^\infty$$

where $K^\infty \setminus K^*$ is pseudoconcave i.e. has local maximum property with respect to the functions $|P|$, $P \in \mathbb{C}[z_1, z_2, \dots, z_n]$. Furthermore, $K^\infty \setminus K^*$ separates Ω.

Proof. First of all we point out the following fact whose proof is a straightforward consequence of the definition. Let $\{X_t\}_{t\in T}$, where T is a (direct) partially ordered set, be a family of relatively closed subsets of an open subset W of $\mathbb{C}^n \times (0,+\infty)$. Assume all X_t have local maximum property relative to $\mathcal{P}_{\mathcal{H}}$. Then

$$\limsup_{t\to+\infty} X_t = \bigcap_{t^\circ} \overline{\bigcup_{t\geq t^\circ} X_t}$$

has local maximum property relative to $\mathcal{P}_{\mathcal{H}}$ provided it is nonempty.

In order to prove that $K^\infty \setminus K^*$ is pseudoconcave let $W = \Omega \times (0,+\infty)$ and u be the solution of the parabolic problem (P). We know that v is uniformly continuous in $\overline{W}$.

Let

$$X = \Big\{(z,t) \in \Omega \times (0,+\infty) : v(z,t) = 0\Big\}$$

and

$$X^h = \Big\{(z,t) \in \Omega \times (0,+\infty) : v^h(z,t) = 0\Big\}$$

where $v^h(z,t) = v(z,t+h)$, $h>0$.

Since the equation $v_t = \mathcal{H}(v)$ is invariant with respect to time shift $t \mapsto t+h$, $h \geq 0$, we obtain that $\Big\{X^h \cap W\Big\}_{h>0}$ is a family of sets with local maximum property relative to $\mathcal{P}_{\mathcal{H}}$ defined above. Let

$$X^\infty = \limsup_{h\to+\infty} X^h.$$

By what observed at the beginning, $X^\infty \cap W$ has local maximum property relative to $\mathcal{P}_\mathcal{H}$ provide $X^\infty \cap W \neq \varnothing$. On the other hand, from

$$X^h \cap (\mathbb{C}^n \times \{t\}) = E_{t+h}(K, K^*) \times \{t\},$$

and

$$\limsup_{h \to +\infty} E_{t+h}(K, K^*) = K^\infty,$$

for each $t > 0$ we deduce that $X^\infty = K^\infty \times (0, +\infty)$ and so the set $(K^\infty \setminus K^*) \times (0, +\infty)$ has local maximum property relative to the class of subsolutions $\mathcal{P}_\mathcal{H}$.

Suppose now that $K^\infty \setminus K^*$ is not a local maximum set relative to the functions $|P|$, $P \in \mathbb{C}[z_1, z_2, \ldots, z_n]$. Then, by [6] there are a point $z^0 \in K^\infty \setminus K^* \subset \Omega$ and a strictly plurisubharmonic function $\varrho \in C^2(B(z^0, r))$, $r > 0$, such that $\varrho(z^0) = 0$ and $\varrho(z) < 0$ for $z \in K^\infty \cap (B(z^0, r) \setminus \{z^0\})$. Choose a small $\varepsilon > 0$ such that the function $\psi(z,t) = \varrho(z) - \varepsilon(t - t^0)^2$ satisfies $\mathcal{H}(\psi) - \psi_t > 0$ in $B(z^0, r) \times (t^0 - r, t^0 + r)$, i.e. $\psi \in \mathcal{P}_\mathcal{H}$ in a neighbourhood of (z^0, t^0). Owing to the properties of ϱ,

$$\psi_{|(K^\infty \setminus K^*) \times (0,+\infty)} = \psi_{|X^\infty \cap W}$$

$$\psi_{|(K^\infty \setminus K^*) \times (0,+\infty)} = \psi_{|X^\infty \cap W}$$

has strict local maximum at (z^0, t^0): contradiction. $\square$

We will need the following general fact.

Proposition 2.10. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain and K, K' disjoint compact subsets of Ω. Let $K \cap \mathrm{b}\,\Omega = K^*$, $K' \cap \mathrm{b}\,\Omega = K'^*$. Then*

$$E_t(K, K^*) \cap E_t(K', K'^*) = \varnothing$$

for every $t > 0$.

Proof. Take a continuous function $g : \overline{\Omega} \to \mathbb{R}$ such that $g^{-1}(0) = K$, $g^{-1}(1) = K'$ and solve the problem

$$\begin{cases} v_t &= \mathcal{H}(v) \text{ in } \Omega \times (0, +\infty) \\ v &= g \text{ on } \left(\overline{\Omega} \times \{0\}\right) \cup (\mathrm{b}\,\Omega \times (0, +\infty)) . \end{cases}$$

Then

$$E_t(K, K^*) = \{v(\cdot, t) = 0\}, \;\; E_t(K', K'^*) = \{v(\cdot, t) = 1\}$$

and consequently the subsets $E_t(K, K^*)$, $E_t(K', K'^*)$ are disjoint for every $t > 0$. $\square$

Remark 2.4. We do not know if the same is true if we have two different strictly pseudoconvex domains Ω, Ω' with $K \subseteq \Omega$, $K' \subseteq \Omega'$.

3. Evolution of graphs

From now on we assume that K is the graph Γ of a continuous function $u : \overline{D} \to \mathbb{R}$ where D is a bounded domain of $\mathbb{C}^{(n-1)} \times \mathbb{R}$ and $K^* = \mathrm{b}\,\Gamma$.

We have the following theorem:

Theorem 3.1. *If $D \times i\mathbb{R}$ is strictly pseudoconvex then $E_t(\Gamma, \mathrm{b}\,\Gamma)$ is a graph for every $t \geq 0$.*

Proof. In our situation $K = \Gamma$ and $D \times i\mathbb{R}$ is a strictly pseudoconvex domain in $\mathbb{C}^n$. Set $z' = (z_1, \ldots, z_{n-1})$ and consider translations $T_h : \mathbb{C}^n \to \mathbb{C}^n$ of the form $(z', z_) \mapsto (z', z_n + ih)$, $h \in \mathbb{R}$.

For fixed $h > 0$, consider a bounded strictly pseudoconvex domain Ω and a large enough number M such that

$$\begin{aligned} \overline{D} \times i\mathbb{R} \supset \overline{\Omega} \;\; &\supset \;\; \overline{D} \times [-iM, iM] \\ &\supset \;\; \Gamma \cup T_h(\Gamma). \end{aligned}$$

Then we can consider the evolutions of Γ and $T_h(\Gamma)$ $(\mathrm{mod}\, \mathrm{b}\,\Gamma)$ with such $\overline{\Omega}$ and they must be disjoint in view of Proposition 2.10. (The evolution is independent of the specific choice of such Ω.) $\square$

The operator $\mathcal{H}$ does not depend upon the equation of a surface. In particular, if $x_1, y_1, \ldots, x_n, y_n$ are real coordinates with $z_\alpha = x_\alpha + iy_\alpha$, $1 \le \alpha \le n$, for a graph of a smooth function $y_n = u(x_1, \ldots, x_n, y_1, \ldots, y_{n-1})$ one has $\mathcal{H}(y_n - u) = \mathcal{H}_0(u)$ where $\mathcal{H}_0$ is a quasilinear degenerate elliptic operator in the real coordinates.

If $n = 2$ $\mathcal{H}_0$ is the Levi operator for graphs (cfr. [7])

$$\begin{aligned}\mathcal{H}_\circ(u) &= \frac{1}{4}(1+|Du|^2)^{-1}\big\{(1+u_3^2)(u_{11}+u_{22}) + (u_1^2+u_2^2)u_{33} \\ &+ 2(u_2 - u_1u_3)u_{13} - 2(u_1+u_2u_3)u_{23}\big\}\end{aligned}$$

$(u_j = \partial u/\partial x_j$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j)$.

Lemma 3.2. *Let u be continuous in a domain $D \subseteq \mathbb{C}^{n-1} \times \mathbb{R}$. Then $y_n - u$ is a weak solution of $v_t = \mathcal{H}(v)$ in $D \times i\mathbb{R} \times (0, +\infty)$ if and only if u is a weak solution of $u_t = \mathcal{H}_0(u)$ in $D \times \mathbb{R}$.*

Proof. Set $x = (x_1, \ldots, x_n)$, $y' = (y_1, \ldots,_{n-1})$. If $v = y_n - u(x, y')$ is a weak solution of $v_t = \mathcal{H}(v)$ in $D \times i\mathbb{R} \times (0, +\infty)$ then is immediately seen that u is a weak solution of $u_t = \mathcal{H}_0(u)$ in $D \times (0, +\infty)$.

Conversely, let us suppose that u is a weak solution of $u_t = \mathcal{H}_0(u)$ and let $\phi = \phi(x, y, t)$ be smooth and such that $y_n - u - \phi$ has a local maximum at $(\bar{x}, \bar{y}', \bar{t})$. We may assume that $(\bar{x}, \bar{y}', \bar{t}) = (0, 0, 0)$ and $u(0, 0, 0) = \phi(0, 0, 0) = 0$. Since, locally at $(0, 0, 0)$, $y_n - u \le \phi$ we have $\phi_{y_n}(0, 0, 0) = 1$. In particular, $\phi = 0$ is a (local) graph $y_n = f(x, y', t)$ and $\phi = \lambda(y_n - h)$ with λ smooth and $\lambda(0, 0, 0) = 1$. Moreover, since $\mathcal{H}$ is invariant with respect to unitary transformations of $\mathbb{C}^n$, we may also assume that $\mathrm{d}_{x,y'} f(0, 0, 0) = 0$. In

this situation we have

$$-u(x,y',t) \le -\lambda(x,0,t)f(x,y',t)$$

and

$$\mathcal{H}_0(-\lambda f)(0) = -\frac{1}{4}\sum_{j=1}^{n_1}\left[f_{x_jx_j}(0,0,0) + f_{y_jy_j}(0,0,0)\right] = \mathcal{H}_0(-f)(0,0,0).$$

Furthermore

$$\phi_t(0,0,0) = f_t(0,0,0),\ \ \mathcal{H}(\phi)(0,0,0) = \mathcal{H}_0(-f)(0,0,0).$$

Since $-u$ is a weak solution of $w_t = \mathcal{H}_0(w)$

$$-f_t(0,0,0) \le \mathcal{H}_(-f)(0,0,0).$$

From this, in view of the above identities, we obtain

$$\phi_t(0,0,0) = -f_t(0,0,0) =\le -\mathcal{H}_0(-f)(0,0,0) = \mathcal{H}_0(-f)(0,0,0) = \mathcal{H}(\phi)(0,0,0).$$

This proves that $y_n - u$ is a weak subsolution.

Similarly we prove that $y_n - u$ is a weak supersolution.

Therefore $v = y_n - u$ is a weak solution of $v_t = \mathcal{H}(v)$. $\square$

Taking into account the semigroup property 4 and independence of defining function we deduce from Lemma 3.2 the following

Lemma 3.3. *Let $v = v(z,t)$ be a local weak solution of $v_t = \mathcal{H}(v)$. Suppose that, locally at (z^0,t^0), $v = 0$ is a graph $y_n = u(x,y',t)$ of a contiunous function. Then u is a weak solution of $u_t = \mathcal{H}_0(u)$.*

Now we are in position to prove the following

Theorem 3.4. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^{n-1}\times\mathbb{R}$, Γ_0 the graph of a continuous function $u_0 : \overline{D} \to \mathbb{R}$. Then the evolution of Γ_0 with fixed boundary*

is governed by the following parabolic problem

$$
\begin{cases}
u_t = \mathcal{H}_0(u) & \textit{in } D \times (0, +\infty) \\
u(x, y', 0) = u_0(x, y') & \text{for } (x, y', 0) \in \overline{D} \times \{0\} \\
u(x, y', t) = u_0(x, y') & \text{for } (x, y', t) \in \mathrm{b}\, D \times [0, +\infty).
\end{cases}
\tag{5}
$$

Proof. Let the evolution be defined by the zero set $\{v = 0\}$ where v is the weak solution of the parabolic problem (P).

In view of Theorem 3.1 every $E_t(\Gamma_0, \mathrm{b}\,\Gamma_0)$, $t \geq 0$, is a graph, a priori over a subset of $\overline{D}$, but in view of Theorem 2.8 it separates $D \times i\mathbb{R}$ so is the graph over $\overline{D}$, say of a continuous function $u^t = u^t(x, y')$. Define $u : \overline{D} \times (0, +\infty) \to \mathbb{R}$ by $u(x, y', t) = u^t(x, y')$. The function u is continuous: if $(x^n, y'^n, t^n) \to (\bar{x}, \bar{y}', \bar{t})$ then the sequence $(x^n, y'^n, u^{t_n}(x^n, y'^n), t^n)$ tends to a point $(\bar{x}, \bar{y}', \bar{y}_n, \bar{t})$ which lies on the graph of u^t. In particular $\bar{y}_n = u(\bar{x}, \bar{y}', \bar{t})$.

Thus

$$E_t(\Gamma_0, \mathrm{b}\,\Gamma_0) = \{y_n = u(x, y', t)\}.$$

Owing to Lemma 3.3 u is a weak solution of $u_t = \mathcal{H}_0(u)$ which satisfies all conditions (5). This concludes the proof. $\square$

The following lemma will be used in the next section

Lemma 3.5. *Let U be a domain in $\mathbb{C}^n$ and $u \in C^0\left(U \times (0, +\infty)\right)$ a continuous subsolution of $u_t = \mathcal{H}_0(u)$ such that $\mathcal{H}_0(u) \leq 0$ (in the weak sense). Then u is non increasing in time.*

Proof. This follows from the more general fact: let $W = V \times (a, b) \subset \mathbb{R}^N$, V open in $\mathbb{R}^{N-1}$, $u = u(x, t)$ an upper semicontinuous function in W such the inequality $u_t \leq 0$ is satisfied in W (in the weak sense). Then, for every $x \in V$, $a < t_1 < t_2 < b$ we have $u(x, t_1) \geq u(x, t_2)$.

Fix t_1 and let

$$W_1 = \left\{(x, t) \in \mathbb{R}^N : x \in V, t_1 < t < b\right\}.$$

We may assume, without loss of generality that $u \leq M < +\infty$ on W_1 (M constant), $m = \inf\ g > -\infty$ and that $b - t_1 < 1$. It suffices to show the following: for every $v \in C^\infty(V)$ such that $u(x, t_1) < v(x)$, it holds $u(x,t) < v(x)$ for every $(x,t) \in W_1$.

Set, for $\alpha \in [0, +\infty)$,

$$\varphi^\alpha(x,t) = v(x) + (M - m)(t - t_1)^\alpha.$$

Then $\varphi^\alpha \in C^\infty(W_1)$ and

$$\varphi^0(x,t) = v(x) + (M - m) \geq u(x,t),$$

$$\lim_{\alpha \to +\infty} \varphi^\alpha(x,t) = v(x),$$

for $t - t_1 < 1$, $(x,t) \in W_1$.

Suppose now that $u(x_0, t_0) > v(x_0)$ for some $(x_0, t_0) \in W_1$. Then there is an $\alpha \in (0, +\infty)$ and $(x^*, t^*) \in W_1$ such that $\varphi^\alpha(x^*, t^*) = u(x^*, t^*)$. Since $u_t \leq 0$ in the weak sense, $\varphi^\alpha((x^*, t^*) \leq 0$, a contradiction. $\square$

4. Limit for solutions

In order to describe the asymptotic behaviour of the weak solution u of (5) we need to recall some results about the existence of Levi flat hypersurfaces with prescribed boundary.

Let $S \subset \mathbb{C}^n$ be a connected smooth submanifold of dimension $(2n-2)$. Assume that:

(1) S is compact and nowhere minimal at its CR points;

(2) S has at least one complex point and every such point of is flat and elliptic;

(3) S does not contain complex manifold of dimension $(n-2)$.

Then in [1] the following two theorems are proved

Theorem 4.1. *S is diffeomorphic to the unit sphere with two complex points p_1, p_2. The CR orbits of S are topological $(2n-3)$-spheres that can be represented as level sets of a*

smooth function $\nu : S \to \mathbb{R}$*, inducing on* $S_0 = S \setminus \{p_1, p_2\}$ *a foliation* $\mathcal{F}$ *of class* C^∞ *with* 1*-codimensional compact leaves.*

Theorem 4.2. *There exist a smooth submanifold* $\widetilde{S}$ *and a Levi-flat* $(2n-1)$*-subvariety* $\widetilde{M}$ *in* $\mathbb{R} \times \mathbb{C}^n$ *(i.e.* $\widetilde{M}$ *is Levi-flat in* $\mathbb{C} \times \mathbb{C}^n$*), both contained in* $[0,1] \times \mathbb{C}^n$*, such that* $\widetilde{S} = d\widetilde{M}$ *in the sense of currents and the natural projection* $\pi \colon [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ *restricts to a diffeomorphism between* $\widetilde{S}$ *and* S*.*

We can go further if S is a graph:

Theorem 4.3. ([2, Theorem 3.1]) *Let* $D \subset \mathbb{C}^{n-1} \times \mathbb{R}$ *be a strictly pseudoconvex bounded domain,* $g_0 : \mathrm{b}\, D \to \mathbb{R}$ *a smooth function. Assume that the graph* S *of* g_0 *satisfies the hypothesis of Theorem 4.2. Then there exists a Lipschitz function* $f : \overline{D} \to \mathbb{R}$ *which is smooth on* $\overline{D} \setminus \{q_1, q_2\}$*, the projections of the only two complex elliptic points of* S*, and such that* $f_{|\mathrm{b}\, D} = g_0$ *and* $M = \mathrm{graph}(f) \setminus S$ *is a Levi flat hypersurface of* $\mathbb{C}^n$*.*

We want to prove that in this situation, the evolution of an arbitrary, continuous graph over D, with boundary S tends as $t \to +\infty$ to the Levi flat graph M. This follows from

Theorem 4.4. *Let* $u_0 : \overline{D} \to \mathbb{R}$ *be a continuous function such that* $g_0 = u_{0|\mathrm{b}\, D}$ *and* $u \in C^0\left(\overline{D} \times [0, +\infty)\right)$ *the weak solution of the problem (5). Then*

$$\lim_{t \to +\infty} u(\cdot, t) = f$$

in $C^0(\overline{D})$*. In particular, if* $\Gamma_0 = \mathrm{graph}(u_0)$ *we have*

$$E_t(\Gamma_0, S) \to M$$

as $t \to +\infty$ *in the* C^0*-topology.*

Proof. Observe that $\mathcal{H}_0(f) = 0$ on D, $M = \mathrm{graph}(f) \setminus S$ being a Levi flat hypersurface. We divide the proof in several steps. First of all we construct two smooth barriers $\delta^{\pm}$: $\delta^- \le u_0 \le \delta^+$ in D, $\delta^- = \delta^+ = u_0$ on $\mathrm{b}\, D$ and $\mathcal{H}_0(\delta^-) \ge 0$, $\mathcal{H}_0(\delta^+) \le 0$ in D. This is easily done using the functions $\delta^{\pm} = u_0 \mp \lambda\varrho$ where λ is a suitable positive constant $\varrho = \varrho(x, y')$ a smooth function on a neighbourhood U of $\overline{D}$ with the following properties: ϱ is strictly plurisubharmonic in $D \times i\mathbb{R}$, $D = \{\varrho < 0\}$ and $d\varrho \ne 0$ on $\mathrm{b}\, D$.

Next we consider the weak solutions $u^{\pm}$ of (5) corresponding respectively to the boundary values $\delta^{\pm}$ on D and g_0 on $\mathrm{b}\, D \times [0, +\infty)$. $u^{\pm}$ are bounded by virtue of the maximum principle and uniformly continuous because of Walsh's Lemma. Moreover, by Lemma 3.5, u^+ (u^-) is non increasing (non decreasing) in t since $\mathcal{H}_0(\delta^+) \le 0$ ($\mathcal{H}_0(\delta^-) \ge 0$). It follows that $\lim_{t \to +\infty} u^{\pm}(\xi, t) := \tilde{u}^{\pm}(\xi)$ exists pointwise.

Now define functions $u_h^{\pm}(\cdot, t) = u^{\pm}(\cdot, t + h)$ for each positive h. These functions are still weak solutions (with different boundary values). Moreover, since $u^{\pm}$ are bounded, the sets $\left\{u_h^{\pm}\right\}_{h \ge 0}$ are equicontinuous and

$$\tilde{u}^{\pm}(x) = \lim_{t \to +\infty} u^{\pm}(x, t) = \lim_{t \to +\infty} u_h^{\pm}(x, t)$$

for every $\xi \in D$. It follows that $\tilde{u}^{\pm}$ are continuous in D, $\tilde{u}^{\pm} = \phi_0$ on $\mathrm{b}D$ and $\mathcal{H}_0(u^{\pm}) = 0$ in D and consequently (by uniqueness) $\tilde{u}^+ = \tilde{u}^- = w$ in D. Consider now the weak solution u of the parabolic problem (5). By virtue of the comparison principle we have

$$u^-(\cdot, t) \le u(\cdot, t) \le u^+(\cdot, t)$$

and from this, letting $t \to +\infty$ we obtain

$$f(\xi) = \lim_{t \to +\infty} u^-(\xi, t) \le \liminf_{t \to +\infty} u(\xi, t) \le \limsup_{t \to +\infty} u(\xi, t) = \lim_{t \to +\infty} u^+(\xi, t) = f(\xi)$$

for every $\xi \in D$, so

$$\liminf_{t \to +\infty} u(\cdot, t) = \limsup_{t \to +\infty} u(\cdot, t) = \lim_{t \to +\infty} u(\cdot, t) = f$$

in $C^0(\overline{D})$. $\square$

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